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ON THE CONTINUITY OF THE HARDY-LITTLEWOOD MAXIMAL FUNCTION

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Abstract. It is concerned with the continuity of the Hardy-Little wood maximal function between the classical Lebesgue spaces or the Orlicz spaces. A new approach to the continuity of the Hardy-Littlewood maximal function is presented through the observation that the continuity is closely related to the existence of solutions for a certain type of first order ordinary differential equations. It is applied to verify the continuity of the Hardy-Littlewood maximal function from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ for $1 \leq q < p < \infty$.

This paper concerns with the Hardy-Littlewood maximal operator M:

$$Mf(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)|dy.$$

Here $B(x,r)$ is the Euclidean ball of radius r centered at x and f is a locally integrable function. This operator plays an important role in studying the differentiability properties of functions, singular integrals, and it also has numerous applications to the theory of partial differential equations.

One of the most well-known theorems for the Hardy-Littlewood maximal operator would be the continuity property from the Lebesgue space $L^p(\mathbb{R}^n)$ into itself, $1 < p \leq \infty$. This paper investigates the continuities of the Hardy-Littlewood maximal function on different Lebesgue spaces especially, from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ for $1 \leq q < p < \infty$. For this, it is pointed out that the continuity is closely related to the existence of solutions for a certain type of ordinary differential equations, which can be stated as follows:

Proposition 1. *Let Ω be a subset of $\mathbb{R}^n$ (possibly the whole space), and α, β be any absolutely continuous increasing functions on* $(0,\infty)$.

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Suppose that there exists a solution $y(\cdot)$ of increasing function on $(0, \infty)$ satisfying the first order ODE

$$y' = \frac{\alpha'(x)}{\beta'(y)(x-y)}, \qquad ' \equiv \frac{d}{dx} \tag{0.1}$$

for almost every $x > 0$ and $y(0+) = 0$, then we have

$$\int_\Omega \alpha(Mf(x))dx \le C \int_\Omega |f(x)|\beta(|f(x)|)dx$$

for some positive constant C independent of f.

The proof is presented at the end of the paper. The classical theory of the first order ODE says that for any x_0 and y_0 with $0 < y_0 < x_0$, the *characteristic* ODE (0.1) has a unique (local) solution that passes through a point (x_0, y_0), and persists unless the graph touches with the line $y = x$ or $\beta'(y) = 0$. In particular, if $\alpha'(x) > 0$ and $\beta'(y) > 0$ for all $x, y > 0$ (with $\beta'(0) = 0$), then the slope $y'(x)$ of the flow $(x, y(x))$ is always positive as far as it stays in the region

$$\{(x, y) \,|\, x, y > 0,\ y < x\},$$

and it blows up when it touches the line segment $y = x$ or $y = 0$. Here is an application of Proposition 1.

Theorem 1. *For $1 \le q < p < \infty$ or $1 < p = q$, we have*

$$\int_{\mathbb{R}^n} |Mf(x)|^q dx \le C \int_{\mathbb{R}^n} |f(x)|^p dx$$

for some positive constant C independent of f.

Proof. It suffices to prove the existence of solution y of increasing function for the characteristic ODE:

$$y' = \frac{1}{2^p} \frac{x^{q-1}}{y^{p-2}(x-y)} \tag{0.2}$$

with the condition $y(1) = \frac{1}{2}$. The choices of the multiplicative constant $\frac{1}{2^p}$ and the point $(x, y) = (1, \frac{1}{2})$ are only for convenience in order to make the flow stay inside the region $R \equiv \{(x, y) : x, y > 0, x < y\}$. When $p = q > 1$, the characteristic ODE (0.2) is a homogeneous equation whose solution is just $y(x) = \frac{x}{2}$ (a singular solution). In this case, the result coincides with the classical continuity of $M : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$. So the essential part of the proof corresponds to the case when $q < p$.

For $x \geq 1$, the solution y_1 of the ODE (0.2) obeys the homogeneous ordinary differential inequality

$$y' \leq \frac{1}{2^p} \frac{x^{p-1}}{y^{p-2}(x-y)}, \qquad y(1) = \frac{1}{2}. \tag{0.3}$$

Therefore we compare this differential inequality (0.3) with the homogeneous ordinary differential equation

$$y' = \frac{1}{2^p} \frac{x^{p-1}}{y^{p-2}(x-y)}, \qquad y(1) = \frac{1}{2} \tag{0.4}$$

to observe that a (local) solution y_1 of (0.2) satisfies

$$y_1(x) \leq y_0(x), \qquad x \geq 1,$$

where y_0 is a unique solution of (0.4). That is, $y_1(x) \leq \frac{x}{2}$ for $x \geq 1$. Therefore the solution of the characteristic ODE (0.2) never blows up on $[1, \infty)$.

Now, we examine non-blow-up-ness of the solution y_1 of the characteristic ODE (0.2) with the terminal condition $y_1(1) = \frac{1}{2}$ on the interval $(0, 1]$. As in the above, solving the homogeneous ordinary differential inequality

$$y' \geq \frac{1}{2^p} \frac{x^{q-1}}{y^{q-2}(x-y)},$$

we have

$$y_1(x) \geq \frac{x}{2}$$

for $0 < x \leq 1$. This says that the graph of $y_1(\cdot)$ does not meet the x-axis on the interval $(0, 1)$. The positivity of the derivative y_1' yields that the solution $y_1(\cdot)$ is an increasing function on the interval $(0, 1)$, and so the graph of $y_1(\cdot)$ does not meet the line $y = x$ on $(0, 1)$. In fact, suppose that the graph of the solution y_1 gets into touch with the straight line $y = x$. Let x_0 be the nearest point from 1 having $y_1(x_0) = x_0$. Then considering the right hand side of (0.2), the slope of $y_1(x)$ goes to infinity and the flow $(x, y_1(x))$ proceeds to the line $y = x$ as x goes to x_0 from the right side, which is impossible for continuous and increasing functions whose graphs are located under the line $y = x$. In all, the graph of y_1 lies between the two straight lines $y = x$ and $y = \frac{x}{2}$, and so $\lim_{x \to 0+} y_1(x) = 0$. The existence of the solution of increasing function on $(0, \infty)$ for (0.1) is now completed. $\square$

Proof of Proposition 1. Let $\Phi : \mathbb{R}^+ \to \mathbb{R}^+$ denote the solution of increasing function satisfying the characteristic ODE (0.1). We define

$f_t(x) \equiv f(x)$ if $|f(x)| > \Phi(t)$, and $f_t(x) = 0$, otherwise. Then from the fact that M is weak $(1,1)$, we have

$$\begin{aligned}\int_\Omega \alpha(Mf(x))dx &\leq \int_0^\infty \alpha'(t)\ \ |\{x \in \Omega \mid Mf_t(x) > t - \Phi(t)\}|\ dt \\ &\leq C\int_0^\infty \frac{\alpha'(t)}{t-\Phi(t)}\left(\int_\Omega |f_t(x)|dx\right)dt \\ &= C\int_\Omega |f(x)|\left(\int_0^{\Phi^{-1}(|f(x)|)} \frac{\alpha'(t)}{t-\Phi(t)}dt\right)dx \\ &= C\int_\Omega |f(x)|\beta(|f(x)|)dx\end{aligned}$$

for some positive constant $C > 0$. The last equality follows from the fact that $\frac{\alpha'(t)}{t-\Phi(t)} = \{\beta(\Phi(t))\}'$. □

Remark 2. *The characteristic ODE (0.1) can be replaced by the first order ordinary differential inequality*

$$y'\beta'(y) \geq \frac{\alpha'(x)}{x-y}.$$

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